Grothendieck standard conjectures, morphic cohomology and Hodge index theorem

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Abstract

Using morphic cohomology, we produce a sequence of conjectures, called morphic conjectures, which terminates at the Grothendieck standard conjecture A. A refinement of Hodge structures is given, and with the assumption of morphic conjectures, we prove a Hodge index theorem. We answer a question of Friedlander and Lawson by assuming the Grothendieck standard conjecture B and prove that the topological filtration from morphic cohomology is equal to the Grothendieck arithmetic filtration for some cases.

1 Introduction

The homotopy groups of the cycle spaces of a complex projective variety X form a set of invariants, called the Lawson homology groups of X(see [3], [17]). To establish a cohomology-like theory, Friedlander and Lawson produced the notion of algebraic cocycle in [7] and defined the morphic cohomology groups of a projective variety to be the homotopy groups of some algebraic cocycle spaces. Furthermore, for a smooth projective variety X, by using their moving lemma (see [8]), they proved a duality theorem between the Lawson homology and morphic cohomology of X. Walker has defined an inductive limit of mixed Hodge structures on the Lawson homology [21] of X, and we extend this to morphic cohomology by using the above duality isomorphisms. Then the images of the morphic cohomology groups of X in its singular cohomology groups under the natural transformations have sub-Hodge structures.

The Grothendieck standard conjectures have various parts (see [11], [15], [16]). For a smooth projective variety X of dimension m, let $C^j(X)$ be the subspace of $H^{2j}(X;\mathbb{Q})$ which is generated by algebraic cycles. By the Hard Lefschetz theorem, cup product with the Lefschetz class L gives isomorphism

$$\begin{tikzcd}
H^{2j}(X;\mathbb{Q}) \arrow[r, "L^{m-2j}", "\cong"'] & H^{2m-2j}(X;\mathbb{Q}) \\
C^j(X) \arrow[u] \arrow[r] & C^{m-j}(x) \arrow[u]
\end{tikzcd}$$

for $j \leq \lfloor \frac{m}{2} \rfloor$. The Grothendieck standard conjecture A (GSCA for short) claims that the restriction of L^{m-2j} also gives an isomorphism between $C^j(X)$ and $C^{m-j}(X)$, or equivalently, the adjoint operator Λ maps $C^{m-j}(X)$ into $C^j(X)$. The Grothendieck standard conjecture B (GSCB for short) says that the adjoint operator Λ is algebraic, i.e., there is a cycle β on $X \times X$ such that

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$\Lambda : H^*(X;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$ is got by lifting a class from X to $X \times X$ by the first projection, cupping with β and taking the image in $H^*(X;\mathbb{Q})$ by the Gysin homomorphism associated to the second projection. For abelian varieties, the GSCB was proved by Lieberman in [19], and we know the GSCB for a smooth variety which is a complete intersection in some projective space and for Grassmannians (see [11]).

In this paper, a sequence of conjectures, called morphic conjectures, is introduced and the GSCA is the last conjecture in this sequence. We show that if the GSCB holds on X, it implies all the morphic conjectures of X. Various equivalent forms of the morphic conjectures are provided. It is well known that the GSCA is equivalent to the statement that numerical equivalence is equal to homological equivalence. We prove an analogous statement for our morphic conjectures in Proposition 4.3. It was proved by Jannsen (see [14]) that the GSCA is equivalent to the semisimplicity of the ring of algebraic correspondences, we do not know if analogous result is true for morphic conjectures. The refinement of the Hodge structures by the images of the morphic cohomology groups of X in its singular cohomology groups, with the assumption of the corresponding morphic conjecture, is compatible with a refinement of the Lefschetz decomposition, and we get a result analogous to the classical Hodge index theorem.

Let us give a brief review of morphic cohomology (see [9]). Throughout this paper, X, Y are smooth complex projective varieties and the dimension of X is m. An effective Y-valued r-cocycle c is an effective $(r+m)$-cycle in $X \times Y$ such that each fibre of c over X is an r-cycle on Y. We denote the group of effective Y-valued r-cocycles by $\mathscr{C}_r(Y)(X)$. The Chow monoid $\mathscr{C}_{r+m}(X \times Y) := \coprod_{d \geq 0} \mathscr{C}_{r+m,d}(X \times Y)$ with topology from the analytic topology of each Chow variety $\mathscr{C}_{r+m,d}(X \times Y)$ is a topological monoid and we give $\mathscr{C}_r(Y)(X) \hookrightarrow \mathscr{C}_{r+m}(X \times Y)$ the subspace topology. Let $Z_r(Y)(X) := \mathscr{C}_r(Y)(X) \times \mathscr{C}_r(Y)(X)/\sim$ be the naive group-completion of $\mathscr{C}_r(Y)(X)$ with the quotient topology where $(a,b) \sim (c,d)$ if and only if $a+d = b+c$, then $Z_r(Y)(X)$ is a topological abelian group. Define the group of algebraic t-cocycles on X to be

$$Z^t(X) := \frac{Z_0(\mathbb{P}^t)(X)}{Z_0(\mathbb{P}^{t-1})(X)}$$

and define the (t,k)-morphic cohomology group to be

$$L^t H^k(X) := \pi_{2t-k} Z^t(X), \quad 0 \leq k \leq 2t$$

the homotopy group of the cocycle space. It is shown in [20, Theorem A.2] that $Z_0(\mathbb{P}^t)(X)$ is a CW-complex, and a similar argument of [20, Theorem A.5] shows that $Z^t(X)$ is a CW-complex. We list some fundamental properties of morphic cohomology which will be used in this paper.

1. There is a natural transformation $\Phi^{t,k} : L^t H^k(X) \longrightarrow H^k(X)$ from morphic cohomology to singular cohomology for any t, k.

2. There is a cup product pairing $L^t H^k(X) \otimes L^r H^s(X) \longrightarrow L^{t+r} H^{k+s}(X)$ which is transformed to the cup product in singular cohomology by the natural transformations.

3. There is a commutative diagram:

$$\begin{tikzcd} L^t H^k(X) \arrow[r, "\mathscr{D}"] \arrow[d, "\Phi"'] & L_{m-t} H_{2m-k}(X) \arrow[d, "\Psi"] \\ H^k(X) \arrow[r, "\mathscr{P}\mathscr{D}"] & H_{2m-k}(X) \end{tikzcd}$$

for $0 \leq t \leq m$ where $\mathscr{D}$ is the Friedlander-Lawson duality isomorphism, $\mathscr{PD}$ is the Poincaré duality isomorphism, and Ψ is the natural transformation from Lawson homology to singular homology.

In [7], Question 9.7, Lawson and Friedlander asked if the map $\Phi^{t,k} : L^tH^k(X) \longrightarrow H^k(X)$ is surjective for X smooth and $t \geq k$. We answer this question in rational coefficients by assuming GSCB. As a consequence, we prove that the topological filtration from morphic cohomology is equal to the Grothendieck arithmetic filtration for some cases. This may open a new way to check the validity of the Grothendieck standard conjectures and the generalized Hodge conjecture.

2 Inductive limit of mixed Hodge structures

We use HS and MHS to abbreviate Hodge structure and mixed Hodge structure respectively. We follow Walker's definition of the inductive limit of mixed Hodge structures (IMHS) in [21].

Definition An IMHS is an inductive system of MHS's $\{H_\alpha, \alpha \in I\}$ where the index set I is countable such that there exist integers $M < N$ so that $W_M((H_\alpha)_\mathbb{Q}) = 0, W_N((H_\alpha)_\mathbb{Q}) = (H_\alpha)_\mathbb{Q}, F^N((H_\alpha)_\mathbb{Q}) = 0$, and $F^M((H_\alpha)_\mathbb{Q}) = (H_\alpha)_\mathbb{Q}$ for all $\alpha \in I$. Equivalently, an IMHS is a triple $(H, W_\bullet, F^\bullet)$, where H is a countable abelian group, $W_\bullet(H_\mathbb{Q})$ and $F^\bullet(H_\mathbb{C})$ are finite filtrations satisfying

$$Gr_n^W(H_\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$$

where

$$H^{p,q} = F^p Gr_{p+q}^W(H_\mathbb{C}) \cap \overline{F}^q Gr_{p+q}^W(H_\mathbb{C}),$$

and such that every finitely generated subgroup of H is contained in a finitely generated subgroup H' so that $(H', W_\bullet|_{H'_\mathbb{Q}}, F^\bullet|_{H'_\mathbb{C}})$ is a MHS. A morphism of IMHS is morphism of filtered systems of MHS's.

It is shown in Proposition 1.4 of [13] that the category of IMHS is abelian. By [21, Theorem 4.1], the Lawson homology groups of a quasi-projective variety have an IMHS.

Definition We endow $L^tH^k(X)$ with an IMHS by making the Friedlander-Lawson duality map $\mathscr{D} : L^tH^k(X) \longrightarrow L_{m-t}H_{2m-k}(X)$ an isomorphism of IMHS.

Proposition 2.1. *The map $\Phi^{t,k} : L^tH^k(X) \longrightarrow H^k(X)$ is a morphism of IMHS and the IMHS on $Im\Phi^{t,k}$ is a sub-HS of $H^k(X)$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} L^tH^k(X) \arrow[r, "\mathscr{D}"] \arrow[d, "\Phi^{t,k}"'] & L_{m-t}H_{2m-k}(X) \arrow[d, "\Psi_{m-t,2m-k}"] \\ H^k(X) \arrow[r, "\mathscr{PD}"] & H_{2m-k}(X) \end{tikzcd}$$

By [21, Theorem 4.1], $\Psi_{m-k,2m-k}$ is a morphism of IMHS. Since $\mathscr{PD}^{-1}$ is a morphism of HS, $\Phi^{t,k} = \mathscr{PD}^{-1} \circ \Psi_{m-t,2m-k} \circ \mathscr{D}$ is a morphism of IMHS. The IMHS on $H^k(X)$ is a HS, therefore the image of $\Phi^{t,k}$ is a sub-HS of $H^k(X)$. $\square$

Definition Define $\widetilde{L}^t H^k(X) = Im\Phi^{t,k}$ and decompose

$$\widetilde{L}^t H^k(X;\mathbb{C}) = \bigoplus_{p+q=k} H_t^{p,q}(X).$$

We define the morphic Hodge numbers of X to be

$$h_t^{p,q}(X) = dim H_t^{p,q}(X).$$

The following result (see [7], Theorem 4.4) says that the image of Φ is contained in a specific range in the Hodge decomposition.

Theorem 2.2. *For a smooth projective variety X there is an inclusion*

$$\widetilde{L}^t H^k(X;\mathbb{C}) \subset \bigoplus_{\substack{p+q=k\\ |p-q|\le 2t-k}} H^{p,q}(X)$$

The space of the most interest to us is $\widetilde{L}^t H^{2t}(X;\mathbb{Q}) = H_t^{t,t}(X;\mathbb{Q}) = H_t^{t,t}(X) \cap H^{t,t}(X;\mathbb{Q})$ which is the space generated by algebraic cycles with rational coefficients where $H^{t,t}(X;\mathbb{Q}) = H^{2t}(X;\mathbb{Q}) \cap H^{t,t}(X)$. We recall that the Hodge conjecture says that $\widetilde{L}^t H^{2t}(X;\mathbb{Q}) = H^{t,t}(X;\mathbb{Q})$.

3 Signatures

Before we proceed to the definition of morphic signatures, we need the following result in which we correct some part of the proof in [9, Theorem 5.8]. We use $\mathbb{F}$ to indicate any one of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and define $L^t H^k(X;\mathbb{F}) = L^t H^k(X) \otimes \mathbb{F}$.

Recall that there is a S-map (see [7, Theorem 5.2]) which makes the following diagram commutes:

$$\begin{tikzcd} L^t H^k(X;\mathbb{F}) \arrow[rr, "S"] \arrow[dr, "\Phi^{t,k}"'] & & L^{t+1} H^k(X;\mathbb{F}) \arrow[dl, "\Phi^{t+1,k}"] \\ & H^k(X;\mathbb{F}) & \end{tikzcd}$$

And the natural transformation $\Phi : L^* H^k(X) \to H^k(X)$ is induced by the map $i' : Z^t(X) \to Map(X, Z_0(\mathbb{C}^t))$ where $Map(X, Z_0(\mathbb{C}^t))$ is the space of continuous maps from X to $Z_0(\mathbb{C}^t)$ with the compact-open topology and i' is the map induced by the inclusion map $i : \mathscr{C}_0(\mathbb{P}^t)(X) \hookrightarrow Map(X, \mathscr{C}_0(\mathbb{P}^t))$. Recall that $H^k(X) = \pi_{2t-k} Map(X, Z_0(\mathbb{C}^t))$.

Proposition 3.1. *For any $t \ge m$, the three maps in the diagram of the S-map are isomorphisms for $0 \le k \le 2m$.*

Proof. Obviously it is enough to prove the statement for $\mathbb{Q}$-coefficients. We have the following commutative diagrams:

$$\begin{tikzcd} Z_0(\mathbb{P}^{t-1})(X) \arrow[r] \arrow[d, "\mathscr{D}"'] & Z_0(\mathbb{P}^t)(X) \arrow[r] \arrow[d, "\mathscr{D}"'] & Z^t(X) \arrow[d, "\mathscr{D}"'] \\ Z_m(X \times \mathbb{P}^{t-1}) \arrow[r] & Z_m(X \times \mathbb{P}^t) \arrow[r] & Z_m(X \times \mathbb{C}^t) \end{tikzcd}$$

By the Friedlander-Lawson duality theorem (see [9], Theorem 3.3), the first two $\mathscr{D}$ are homotopy equivalences, and by [20, Proposition 3.2], the upper and lower rows induce long exact sequences of homotopy groups

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k Z_0(\mathbb{P}^{t-1})(X) \arrow[r] \arrow[d] & \pi_k Z_0(\mathbb{P}^t)(X) \arrow[r] \arrow[d] & \pi_k Z^t(X) \arrow[r] \arrow[d] & \pi_{k-1} Z_0(\mathbb{P}^{t-1})(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_k Z_m(X \times \mathbb{P}^{t-1}) \arrow[r] & \pi_k Z_m(X \times \mathbb{P}^t) \arrow[r] & \pi_k Z_m(X \times \mathbb{C}^t) \arrow[r] & \pi_{k-1} Z_m(X \times \mathbb{P}^{t-1}) \arrow[r] & \cdots
\end{tikzcd}$$

By the five-lemma we have $\mathscr{D}_* : \pi_k Z^t(X) \xrightarrow{\cong} \pi_k Z_m(X \times \mathbb{C}^k)$ for $k \geq 0$, then by the Whitehead theorem, we know that $Z^t(X)$ is homotopy equivalent to $Z_m(X \times \mathbb{C}^t)$.

If $t \geq m$, $Z_m(X \times \mathbb{C}^t) \cong Z_0(X \times \mathbb{C}^{t-m})$ by the homotopy property of trivial bundles (see [6, Proposition 2.3]). Applying [20, Proposition 3.2] to the following sequence:

$$Z_0(X \times \mathbb{P}^{t-1}) \xrightarrow{i} Z_0(X \times \mathbb{P}^t) \longrightarrow Z_0(X \times \mathbb{A}^{t-m})$$

we get a long exact sequence of homotopy groups,

$$\cdots \longrightarrow \pi_k Z_0(X \times \mathbb{P}^{t-m-1}) \xrightarrow{i_*} \pi_k Z_0(X \times \mathbb{P}^{t-m}) \longrightarrow \pi_k Z_0(X \times \mathbb{A}^{m-t}) \longrightarrow \pi_{k-1} Z_0(X \times \mathbb{P}^{t-m-1}) \longrightarrow \cdots$$

Recall that the Dold-Thom theorem ([2, 6.10]) says that for a CW-complex A, $\pi_k Z_0(A) \cong H_k^{BM}(A; \mathbb{Z})$ for all k. Applying the Dold-Thom theorem and tensor by $\mathbb{Q}$, we get a long exact sequence:

$$\cdots \longrightarrow H_k(X \times \mathbb{P}^{t-m-1}; \mathbb{Q}) \xrightarrow{i_*} H_k(X \times \mathbb{P}^{t-m}; \mathbb{Q}) \longrightarrow H_k^{BM}(X \times \mathbb{A}^{m-t}; \mathbb{Q})$$
$$\longrightarrow H_{k-1}(X \times \mathbb{P}^{t-m-1}; \mathbb{Q}) \longrightarrow \cdots$$

where i_* is induced from the inclusion map $i : X \times \mathbb{P}^{t-m-1} \subset X \times \mathbb{P}^{t-m}$. Since the inclusion map $j : \mathbb{P}^{t-m-1} \subset \mathbb{P}^{t-m}$ induces an isomorphism $j_* : H_k(\mathbb{P}^{t-m-1}) \longrightarrow H_k(\mathbb{P}^{t-m})$ in homology groups for $0 \leq k \leq 2(t-m-1)$, by the Künneth formula in homology for $H_k(X \times \mathbb{P}^{t-m-1}; \mathbb{Q})$ and $H_k(X \times \mathbb{P}^{t-m}; \mathbb{Q})$, it is not difficult to see that

$$H_k^{BM}(X \times \mathbb{C}^{t-m}; \mathbb{Q}) = \begin{cases} 0, & \text{if } k < 2(t-m) \\ H_{k-2(t-m)}(X; \mathbb{Q}), & \text{if } k \geq 2(t-m) \end{cases}$$

Therefore, if $t \geq m$, since all maps in the chain of isomorphisms

$$L^t H^k(X; \mathbb{Q}) \cong \pi_{2t-k} Z_0(X \times \mathbb{C}^{t-m}) \otimes \mathbb{Q} \cong H_{2t-k}^{BM}(X \times \mathbb{C}^{t-m}; \mathbb{Q}) \cong H_{2m-k}(X; \mathbb{Q}) \cong H^k(X; \mathbb{Q})$$

are natural, their composite is $\Phi^{t,k}$. And from the commutative diagram of the S-map, we see that the S-map is also an isomorphism. $\square$

Suppose that $(\, , \,)$ is a symmetric bilinear form on a finite dimensional vector space V over $\mathbb{Q}$. The signature of $(\, , \,)$ is the number of positive eigenvalues minus the number of negative eigenvalues in a matrix representation of $(\, , \,)$.

From the natural transformation $\Phi^{t,k} \otimes \mathbb{F} : L^t H^k(X; \mathbb{F}) \longrightarrow H^k(X; \mathbb{F})$, we define

$$\widetilde{L}^t H^k(X; \mathbb{F}) = Im(\Phi^{t,k} \otimes \mathbb{F}) \subset H^k(X; \mathbb{F}).$$

Definition (morphic signatures) Suppose the dimension of X is $m = 2n$. For $t \geq n$, we define the t-th morphic signature of X, denoted by σ_t, to be the signature of the symmetric bilinear form:

$$(\ , \ ) : \widetilde{L}^t H^m(X;\mathbb{Q}) \otimes \widetilde{L}^t H^m(X;\mathbb{Q}) \longrightarrow \widetilde{L}^{2t} H^{2m}(X;\mathbb{Q}) = \mathbb{Q}.$$

For $t = m$, since $\widetilde{L}^m H^m(X;\mathbb{Q}) = H^m(X;\mathbb{Q})$ and the cup product in morphic cohomology in this case is just the usual cup product of singular cohomology, σ_m is the usual signature of X. So we have a sequence of signatures $\sigma_m, \sigma_{m-1}, ..., \sigma_n$ which reveals more and more algebraic information of X.

4 The Morphic Conjectures

Let a, b be two nonnegative integers. Define

$$EH^a(X;\mathbb{F}) = \widetilde{L}^a H^0(X;\mathbb{F}) \oplus \widetilde{L}^{a+1} H^2(X;\mathbb{F}) \oplus \cdots \oplus \widetilde{L}^{a+m} H^{2m}(X;\mathbb{F}),$$

$$OH^b(X;\mathbb{F}) = \widetilde{L}^b H^1(X;\mathbb{F}) \oplus \widetilde{L}^{b+1} H^3(X;\mathbb{F}) \oplus \cdots \oplus \widetilde{L}^{b+m-1} H^{2m-1}(X;\mathbb{F})$$

where E and O stand for even and odd respectively. In particular, $EH^0(X;\mathbb{Q})$ is the ring of rational algebraic cohomology classes on X. Define

$$LH^{a,b}(X;\mathbb{F}) = EH^a(X;\mathbb{F}) \oplus OH^b(X;\mathbb{F}).$$

Let $\Omega \in L^1H^2(X)$ be a class coming from a hyperplane section on X. Define an operation

$$\mathcal{L} : L^t H^k(X) \longrightarrow L^{t+1} H^{k+2}(X)$$

by $\mathcal{L}(\alpha) = \Omega \cdot \alpha$. Under the transformation $\Phi^{*,*}$, $\mathcal{L}$ carries over to the standard Lefschetz operator L. The operators induced by $\mathcal{L}$ on $EH^a(X;\mathbb{F}), OH^b(X;\mathbb{F})$ and $LH^{a,b}(X;\mathbb{F})$ are simply the restriction of L to these spaces, and these spaces are L-invariant. By abuse of notation, we use $\mathcal{L}$ to denote the restriction of L to these spaces.

Recall that there is a standard Hermitian inner product on $\mathscr{A}^{p,q}(X)$, the (p,q)-forms on X, called the Hodge inner product defined by

$$< \alpha, \beta >= \int_X \alpha \wedge *\bar{\beta}$$

where $*$ is the Hodge star operator. Let Λ be the adjoint of L with respect to the Hodge inner product. Since L, Λ commute with the Laplacian, they can be defined on the harmonic spaces. From the Hodge theorem we know that the (p,q)-cohomology group of X is isomorphic to the space of (p,q)-harmonic forms. The Hodge inner product induces a Hermitian inner product in harmonic spaces which we also call the Hodge inner product. Restrict the Hodge inner product to $EH^a(X;\mathbb{F}), OH^b(X;\mathbb{F})$ and $LH^{a,b}(X;\mathbb{F})$ respectively and let λ be the adjoint of $\mathcal{L}$ with respect to the Hodge inner product.

Conjecture (morphic conjectures) The morphic conjecture on $EH^a(X;\mathbb{F}), OH^b(X;\mathbb{F})$ and $LH^{a,b}(X;\mathbb{F})$ respectively is the assertion that λ is the restriction of Λ on them respectively.

It is not difficult to see that if a morphic conjecture holds for $\mathbb{Q}$-coefficients, it also holds for $\mathbb{R}$- and $\mathbb{C}$-coefficients. So most of the time we will only work with $\mathbb{Q}$-coefficients.

Definition Consider the cup product pairing:

$$L^tH^k(X;\mathbb{Q}) \otimes L^{t+m-k}H^{2m-k}(X;\mathbb{Q}) \longrightarrow L^{2t+m-k}H^{2m}(X;\mathbb{Q})$$

For $\alpha \in L^tH^k(X;\mathbb{Q})$, we say that α is morphic numerically equivalent to 0 if $\alpha \wedge \beta = 0$ for all $\beta \in L^{t+m-k}H^{2m-k}(X;\mathbb{Q})$. The class α is said to be morphic homologically equivalent to 0 if $\Phi(\alpha) = 0$ where $\Phi : L^tH^k(X;\mathbb{Q}) \longrightarrow H^k(X;\mathbb{Q})$ is the natural transformation. We use MNE for morphic numerical equivalence and MHE for morphic homological equivalence.

Let $Alg_k(X;\mathbb{Q})$ be the group of k-cycles with rational coefficients on X quotient by algebraic equivalence and let $Alg^k(X;\mathbb{Q}) = Alg_{m-k}(X;\mathbb{Q})$. We recall that a class $\alpha \in Alg^k(X;\mathbb{Q})$ is said to be numerically equivalent to zero if $\alpha \bullet \beta = 0$ for all $\beta \in Alg^{m-k}(X;\mathbb{Q})$ where $\bullet$ is the intersection product and α is said to be homologically equivalent to zero if under the cycle map $\gamma : Alg^k(X;\mathbb{Q}) \longrightarrow H^{2k}(X;\mathbb{Q})$, α is sent to zero (see e.g [15]). By the Friedlander-Lawson duality theorem, we can identify $L^tH^{2t}(X;\mathbb{Q})$ with $Alg^t(X;\mathbb{Q})$ for $0 \le t \le m$ (see [9, Theorem 5.1]).

From what we explain before, we have the following result.

Proposition 4.1. *1. On $\oplus_{t=0}^m L^tH^{2t}(X;\mathbb{Q})$ morphic numerical equivalence is same as numerical equivalence and morphic homological equivalence is same as homological equivalence.*

2. If α is morphic homologically equivalent to zero, then α is morphic numerically equivalent to zero.

Proposition 4.2. *1. $dim\widetilde{L}^{a+t}H^{2t}(X;\mathbb{Q}) \le dim\widetilde{L}^{a+m-t}H^{2m-2t}(X;\mathbb{Q})$ for $t \le \lfloor \frac{m}{2} \rfloor$*

2. $dim\widetilde{L}^{b+t}H^{2t-1}(X;\mathbb{Q}) \le dim\widetilde{L}^{b+m-t+1}H^{2m-2t+1}(X;\mathbb{Q})$ for $t \le \lfloor \frac{m}{2} \rfloor$

Proof. We have the following commutative diagrams:

$$\begin{array}{ccc} L^{a+t}H^{2t} & \xrightarrow{\mathcal{L}^{m-2t}} & L^{a+m-t}H^{2m-2t} \\ \downarrow & & \downarrow \\ H^{2t} & \xrightarrow{L^{m-2t}} & H^{2m-2t} \end{array} \qquad \begin{array}{ccc} L^{b+t}H^{2t-1} & \xrightarrow{\mathcal{L}^{m-2t+1}} & L^{b+m-t+1}H^{2m-2t+1} \\ \downarrow & & \downarrow \\ H^{2t-1} & \xrightarrow{L^{m-2t+1}} & H^{2m-2t+1} \end{array}$$

By the Hard Lefschetz theorem, L^{m-2t} and L^{m-2t+1} are isomorphisms for $t \le \lfloor \frac{m}{2} \rfloor$, so we have the conclusions. $\square$

Let $\mathscr{A}$ be any one of $EH^a(X;\mathbb{Q}), OH^b(X;\mathbb{Q})$ and $LH^{a,b}(X;\mathbb{Q})$. Let $\widetilde{\mathscr{A}}$ be the direct sum of all morphic cohomology group $L^tH^k(X;\mathbb{Q})$ such that $\widetilde{L}^tH^k(X;\mathbb{Q})$ is a direct summand of $\mathscr{A}$.

Proposition 4.3. *The followings are equivalent:*

1. *MNE=MHE on $\widetilde{\mathscr{A}}$.*
2. *$dim\widetilde{L}^tH^k(X;\mathbb{Q}) = dim\widetilde{L}^{t+m-k}H^{2m-k}(X;\mathbb{Q})$ for $\widetilde{L}^tH^k(X;\mathbb{Q}) \subset \mathscr{A}$.*
3. *If $\alpha \in \widetilde{L}^tH^k(X;\mathbb{Q}) \subset \mathscr{A}$ for $k \le m$ and $\alpha = \sum_{r\ge 0} L^r\alpha_r$ is the Lefschetz decomposition of α, then $\alpha_r \in \widetilde{L}^{t-r}H^{k-2r}(X;\mathbb{Q}) \subset \mathscr{A}$, for $r \ge 0$.*
4. *If $\alpha \in \mathscr{A}$ then $*\alpha \in \mathscr{A}$.*
5. *If $\alpha \in \mathscr{A}$ then $\Lambda\alpha \in \mathscr{A}$.*

6. The morphic conjecture holds on $\mathscr{A}$.

Proof. We are going to show that $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 2$ and $4 \to 1$.

$1 \to 2$: We consider only the case $\mathscr{A} = EH^a(X;\mathbb{Q})$ since similar argument applies for $\mathscr{A} = OH^b(X;\mathbb{Q})$ and thus for $\mathscr{A} = LH^{a,b}(X;\mathbb{Q})$. For $t \leq \lfloor \frac{m}{2} \rfloor$, consider the commutative diagram

$$\begin{array}{ccc} L^{a+t}H^{2t}(X;\mathbb{Q}) \otimes L^{a+m-t}H^{2m-2t}(X;\mathbb{Q}) & \longrightarrow & L^{2a+m}H^{2m}(X;\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^{2t}(X;\mathbb{Q}) \otimes H^{2m-2t}(X;\mathbb{Q}) & \longrightarrow & H^{2m}(X;\mathbb{Q}) \end{array}$$

If $\widetilde{L}^{a+m-t}H^{2m-2t}(X;\mathbb{Q}) = 0$, by the proposition above, $\widetilde{L}^{a+t}H^{2t}(X;\mathbb{Q}) = 0$. So we may assume that $\widetilde{L}^{a+m-t}H^{2m-2t}(X;\mathbb{Q}) \neq 0$. Let $\alpha \in L^{a+m-t}H^{2m-2t}(X;\mathbb{Q})$ such that $\Phi(\alpha) \neq 0$. If MNE=MHE on $\widetilde{\mathscr{A}}$, then there is a $\beta \in L^{a+t}H^{2t}(X;\mathbb{Q})$ such that $(\beta, \alpha) \neq 0$ where $(\ ,\ )$ is the cup product pairing in morphic cohomology. Thus $(\Phi(\beta), \Phi(\alpha)) \neq 0$. Consequently, the cup product pairing $(\ ,\ )$ in singular cohomology restricted to $\widetilde{L}^{a+t}H^{2t}(X;\mathbb{Q}) \otimes \widetilde{L}^{a+m-t}H^{2m-2t}(X;\mathbb{Q})$ is nondegenerate. It follows that $dim\widetilde{L}^{a+t}H^{2t}(X;\mathbb{Q}) = dim\widetilde{L}^{a+m-t}H^{2m-2t}(X;\mathbb{Q})$.

$2 \to 3$: From the commutative diagram

$$\begin{array}{ccc} L^{t-1}H^{k-2}(X;\mathbb{Q}) & \xrightarrow{\mathcal{L}^{m-k+2}} & L^{t+m-k+1}H^{2m-k+2}(X;\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^{k-2}(X;\mathbb{Q}) & \xrightarrow{L^{m-k+2}} & H^{2m-k+2}(X;\mathbb{Q}) \end{array}$$

we see that L^{m-k+2} maps $\widetilde{L}^{t-1}H^{k-2}(X;\mathbb{Q})$ injectively into $\widetilde{L}^{t+m-k+1}H^{2m-k+2}(X;\mathbb{Q})$. The assumption $dim\widetilde{L}^{t-1}H^{k-2}(X;\mathbb{Q}) = dim\widetilde{L}^{t+m-k+1}H^{2m-k+2}(X;\mathbb{Q})$ implies that L^{m-k+2} restricted to $\widetilde{L}^{t-1}H^{k-2}(X;\mathbb{Q})$ is an isomorphism. Let $\alpha = \sum_{r\geq 0} L^r\alpha_r \in \widetilde{L}^tH^k(X;\mathbb{Q})$ be the Lefschetz decomposition of α. We prove by induction on the length of the Lefschetz decomposition. Since $L^{m-k+2}(\sum_{r\geq 1} L^{r-1}\alpha_r) = L^{m-k+1}(\alpha) \in \widetilde{L}^{t+m-k+1}H^{2m-k+2}(X;\mathbb{Q})$, we have $\sum_{r\geq 1} L^{r-1}\alpha_r \in \widetilde{L}^{t-1}H^{k-2}(X;\mathbb{Q})$. By induction hypothesis, $\alpha_r \in \widetilde{L}^{t-r}H^{k-2r}(X;\mathbb{Q})$ for $r \geq 1$. But $\alpha_0 = \alpha - L(\sum_{r\geq 1} L^{r-1}\alpha_r) \in \widetilde{L}^tH^k(X;\mathbb{Q})$. This completes the proof.

$3 \to 4$: Suppose that $\alpha \in \widetilde{L}^tH^k(X;\mathbb{Q})$ and $\alpha = \sum_{r\geq 0} L^r\alpha_r$ is the Lefschetz decomposition of α. By some calculation we get $*L^j\beta = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(m-k-j)!} L^{m-k-j}\beta$ for $\beta \in B^k$ as a formula in Definition 6. From the assumption $\alpha_r \in \mathscr{A}$, we have $L^{m-k+r}\alpha_r \in \mathscr{A}$, for $r \geq 0$. Thus $*\alpha \in \mathscr{A}$.

$4 \to 5$: From the formula $\Lambda = *L*$ as in Definition 6, we have the conclusion immediately.

$5 \to 6$: Since $\lambda = \pi \circ \Lambda$ where $\pi : \oplus_{k=0}^{2m} H^k(X;\mathbb{Q}) \longrightarrow \mathscr{A}$ is the projection, from the assumption $\pi \circ \Lambda|_{\mathscr{A}} = \Lambda|_{\mathscr{A}}$, we have $\lambda = \Lambda|_{\mathscr{A}}$. Therefore, the morphic conjecture holds on $\mathscr{A}$.

$6 \to 2$: By the Hard Lefschetz theorem, $\Lambda^{m-k} : H^{2m-k}(X;\mathbb{Q}) \longrightarrow H^k(X;\mathbb{Q})$ is an isomorphism for $k \leq m$. Therefore if $\lambda = \Lambda|_{\mathscr{A}}$, $\Lambda^{m-k}(\widetilde{L}^{t+m-k}H^{2m-k}(X;\mathbb{Q})) \subset \widetilde{L}^tH^k(X;\mathbb{Q})$ which implies that they have the same dimension.

$4 \to 1$: Suppose that $\alpha \in L^tH^k(X;\mathbb{Q})$ is morphic numerically equivalent to zero. If $\Phi(\alpha) \neq 0$ then

$$(\Phi(\alpha), *\Phi(\alpha)) = \int_X \Phi(\alpha) \wedge *\Phi(\alpha) =< \Phi(\alpha), \Phi(\alpha) >\neq 0$$

But by the hypothesis $*\Phi(\alpha) \in \widetilde{L}^{t+m-k}H^{2m-k}(X;\mathbb{Q})$, so we can find $\beta \in L^{t+m-k}H^{2m-k}(X;\mathbb{Q})$ such that $\Phi(\beta) = *\Phi(\alpha)$. Then $(\alpha, \beta) = (\Phi(\alpha), \Phi(\beta)) \neq 0$ which contradicts to the assumption. Thus α is morphic homologically equivalent to zero. $\square$

In particular, GSCA is equivalent to the morphic conjecture on $EH^0(X)$.

Definition For $\beta \in L^rH^{2r}(X\times X;\mathbb{Q})$, β induces a map

$$\beta_*: L^tH^k(X;\mathbb{Q}) \longrightarrow L^{t-(m-r)}H^{k-2(m-r)}(X;\mathbb{Q})$$

defined by

$$\beta_*(\alpha) = \mathscr{D}^{-1}(q_*(\mathscr{D}(p^*\alpha)\bullet \mathscr{D}(\beta)))$$

where $p,q: X\times X \longrightarrow X$ are the projections to the first and second factor respectively, $\mathscr{D}$ is the Friedlander-Lawson duality map and $\bullet$ is the intersection product in Lawson homology. An endomorphism $f:\mathscr{A}\longrightarrow\mathscr{A}$ is said to be algebraic if there is a $\beta\in\oplus_{r=0}^m L^rH^{2r}(X\times X;\mathbb{Q})$ such that $\beta_* = f$.

We note that this definition is equivalent to the definition in [11].

Proposition 4.4. *If the Grothendieck standard conjecture B holds on X, then it implies all the equivalent statements in Proposition 4.3 for $\mathscr{A}$ equals to any of $EH^a(X;\mathbb{Q}), OH^b(X;\mathbb{Q})$ or $LH^{a,b}(X;\mathbb{Q})$.*

Proof. If the GSCB holds on X, then Λ is an algebraic operator, thus there exists a cycle $\beta\in L^{m-1}H^{2(m-1)}(X\times X;\mathbb{Q})$ such that $\Lambda=\beta_*$. For $L^tH^k(X;\mathbb{Q})$ a direct summand of $\widetilde{\mathscr{A}}$, $\beta_*(L^tH^k(X;\mathbb{Q}))\subset L^{t-1}H^{k-2}(X;\mathbb{Q})$, thus Λ is an endomorphism of $\mathscr{A}$. By the fifth statement in Proposition 4.3, the morphic conjecture holds on $\mathscr{A}$. □

Hence all the morphic conjectures are true for abelian varieties, varieties of complete intersection and Grassmannians.

By assuming the GSCB, we answer a question of Friedlander and Lawson in rational coefficients (see [7], Question 9.7).

Theorem 4.5. *If the Grothendieck standard conjecture B holds on X, then the map*

$$\Phi^{t,k}: L^tH^k(X;\mathbb{Q}) \longrightarrow H^k(X;\mathbb{Q})$$

is surjective whenever $t\geq k$.

Proof. By Proposition 3.1, it is true if $t\geq m$. So we assume that $t<m$. Then $k<m$. If the GSCB holds on X, we have all the morphic conjectures. Thus from Proposition 4.3, the dimension of $\widetilde{L}^{m+t-k}H^{2m-k}(X;\mathbb{Q})$ is same as the dimension of $\widetilde{L}^tH^k(X;\mathbb{Q})$. Since $m+t-k\geq m$, $\widetilde{L}^{m+t-k}H^{2m-k}(X;\mathbb{Q}) = H^{2m-k}(X;\mathbb{Q})$, and by the Hard Lefschetz theorem, we have $\widetilde{L}^tH^k(X;\mathbb{Q}) = H^k(X;\mathbb{Q})$. Therefore $\Phi^{t,k}$ is surjective. □

5 Topological Filtration And Arithmetic Filtration

In [10], Friedlander and Mazur defines two filtrations on the singular homology groups of a projective variety. One is formed by taking the images of the natural transformations from Lawson homology to singular homology and the other one is the homological version of the Grothendieck's arithmetic filtration. The first filtration is called the topological filtration (denoted by $T_rH_n(X;\mathbb{Q})$) and the second one is called the geometric filtration (denoted by $G_rH_n(X;\mathbb{Q})$). Friedlander and Mazur conjecture that these two filtrations are equal.

Conjecture (Friedlander-Mazur) Let j, n be non-negative integers. For any smooth projective variety X,

$$T_j H_n(X;\mathbb{Q}) = G_j H_n(X;\mathbb{Q})$$

In the following, we define a filtration from morphic cohomology and reformulate the Friedlander-Mazur conjecture as an equality between this filtration and the Grothendieck's arithmetic filtration.

For a variety Y, let $\gamma : \widetilde{Y} \longrightarrow Y$ be a desingularization of Y. Recall that the arithmetic filtration (coniveau filtration) $\{N^pH^*(X;\mathbb{Q})\}_{p\geq 0}$ of $H^*(X;\mathbb{Q})$ is given by

$$N^pH^l(X;\mathbb{Q}) = \{ \text{ Gysin images } \gamma_* : H^{l-2q}(\widetilde{Y};\mathbb{Q}) \longrightarrow H^l(X;\mathbb{Q}) | Y \subset X, \text{ codim }_X Y = q \text{ (pure) }, q \geq p\}$$

(see [18], page 87 for details); and recall that the niveau filtration $\{N_pH_*(X;\mathbb{Q})\}_{p\geq 0}$ of $H_*(X;\mathbb{Q})$ is defined by

$$N_pH_i(X;\mathbb{Q}) = \text{ span } \{ \text{ images } i_* : H_i(Y;\mathbb{Q}) \longrightarrow H_i(X;\mathbb{Q}) | i : Y \hookrightarrow X, \text{ dim Y} \leq p\}$$

Define the topological filtration $\{T^pH^*(X;\mathbb{Q})\}$ to be

$$T^pH^l(X;\mathbb{Q}) = \{ \text{ images } \Phi^{p,l} : L^pH^1(X;\mathbb{Q}) \longrightarrow H^l(X;\mathbb{Q})\}$$

where $\Phi^{p,l}$ is the natural transformation from morphic cohomology to singular cohomology.

If X is a smooth projective variety of dimension m, it is not difficult to see that $G_rH_n(X;\mathbb{Q}) = N_{n-r}H_n(X;\mathbb{Q})$ and $N_pH_i(X;\mathbb{Q}) \cong N^{m-p}H^{2m-i}(X;\mathbb{Q})$ by the Poincaré duality. By [9, Theorem 5.9], $T^rH^n(X;\mathbb{Q}) \cong T_{m-r}H_{2m-n}(X;\mathbb{Q})$. It is proved in [10, 7.5, Corollary 3], that $T_rH_n(X;\mathbb{Q}) \subset G_rH_n(X;\mathbb{Q})$. The cohomological version of this result is the containment $T^{l-p}H^l(X;\mathbb{Q}) \subset N^pH^l(X;\mathbb{Q})$ and the cohomological version of the Friedlander-Mazur conjecture is the following conjecture.

Conjecture For nonnegative integers l, p, $T^{l-p}H^l(X;\mathbb{Q}) = N^pH^l(X;\mathbb{Q})$.

Recall that the generalized Hodge conjecture is the assertion that $N^pH^l(X;\mathbb{Q}) = F_h^pH^l(X;\mathbb{Q})$ for all p, l where $\{F_h^PH^*(X;\mathbb{Q})\}$ is the rational Hodge filtration (see [18], page 88). If the Friedlander-Mazur conjecture holds, ideally, it would give a more concrete picture about the arithmetic filtration.

As a consequence of Theorem 4.5, we have some evidence for the Friedlander-Mazur conjecture.

Corollary 5.1. *If the Grothendieck standard conjecture B holds on a smooth projective variety X. Then*

$$T^tH^k(X;\mathbb{Q}) = N^0H^k(X;\mathbb{Q}) = F_h^0H^k(X;\mathbb{Q}) = H^k(X;\mathbb{Q})$$

for $t \geq k$.

In the following, we give a simple proof of the Friedlander's result in [5, Proposition 4.2].

Proposition 5.2. *Suppose that X is a smooth projective variety. If the Grothendieck standard conjecture B is valid for a resolution of singularities of each irreducible subvariety $Y \subset X$ of codimension $\geq p$. Then*

$$N^pH^l(X;\mathbb{Q}) = T^{l-p}H^l(X;\mathbb{Q})$$

Proof. Suppose that $Y \subset X$ is a subvariety of codimension $p' \geq p$. Let $\sigma : \widetilde{Y} \longrightarrow Y$ be a desingularization and the GSCB holds on $\widetilde{Y}$. Consider the following commutative diagram:

$$\begin{tikzcd} L^{l-p-p'}H^{l-2p'}(\widetilde{Y};\mathbb{Q}) \arrow[r, "\sigma_*"] \arrow[d, "\Phi^{l-p-p',l-2p'}"'] & L^{l-p}H^l(X;\mathbb{Q}) \arrow[d, "\Phi^{l-p,l}"] \\ H^{l-2p'}(\widetilde{Y};\mathbb{Q}) \arrow[r, "\sigma_*"] & H^l(X;\mathbb{Q}) \end{tikzcd}$$

By Theorem 4.5, $\Phi^{l-p-p',l-2p'}$ is surjective. Therefore the image of σ_* is contained in the image of $\Phi^{l-p,l}$. Therefore, $N^pH^l(X;\mathbb{Q}) \subset T^{l-p}H^l(X)$. $\square$

Since the GSCB holds for smooth projective varieties of dimension ≤ 2, we have the following result.

Corollary 5.3. *Suppose that X is a smooth projective variety of dimension less than or equal to 3. Then $N^pH^l(X) = T^{l-p}H^l(X)$ for all p, l.*

6 Abstract Hodge index theorem

Definition Let $V = \oplus_{k=0}^{2m} H^k$ where each H^k is a finite dimensional vector space over $\mathbb{Q}$ and let $V_{\mathbb{F}} = V \otimes \mathbb{F}$, $H^k_{\mathbb{F}} = H^k \otimes \mathbb{F}$. Let $h = \sum_{k=0}^{2m}(k-m)\pi_k$ where $\pi_k : V_{\mathbb{C}} \to H^k_{\mathbb{C}}$ is the projection. V is called a Lefschetz algebra if

1. there is an inner product $<,>: V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ which induces a hermitian inner product $<,>: V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ defined by $< a \otimes \mu, b \otimes \lambda > := \mu\overline{\lambda} < a, b >$.
2. There is an endomorphism $L : V \to V$ of degree 2 with adjoint Λ such that L, Λ and h define a $sl_2(\mathbb{C})$-action on $V_{\mathbb{C}}$ in the following way:

$$[\Lambda, L] = h, [h, \Lambda] = 2\Lambda, [h, L] = -2L$$

Definition Let $B^k := ker\Lambda : H^k_{\mathbb{C}} \to H^{k-1}_{\mathbb{C}}$ be the primitive space. For $\alpha \in B^k$, define

$$\overline{*}L^j\alpha := (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(m-k-j)!} L^{m-k-j}\overline{\alpha}$$

and define

$$\Lambda L^j\alpha := j(m-k-j+1)L^{j-1}\alpha$$

Proposition 6.1. *For a Lefschetz algebra V as above, we have the following properties:*

1. *There is a Strong Lefschetz theorem:*

$$L^{m-k} : H^k \xrightarrow{\cong} H^{2m-k}$$

2. *There is a Lefschetz decomposition: for $a \in H^k$,*

$$a = \sum_{j \geq max(0,k-m)} L^j\alpha_j$$

where $\alpha_j \in B^{k-2j}$.

3. *The Lefschetz decomposition is orthogonal with respect to* $<,>$.

4. $\overline{*}^2 = id$, $\overline{*}$ *is conjugate self-adjoint, i.e.,* $< \alpha, \overline{*}\beta >= \overline{< \overline{*}\alpha, \beta >}$.

5. $\Lambda = \overline{*}L\overline{*}$.

Proof. (1) and (2) follow from the properties of $sl_2(\mathbb{C})$-action (see [18, Theorem 11.15]). Let $L^k\alpha \in L^kB^{n-2k}, L^s\beta \in L^sB^{n'-2s}$ and $k \geq s$. By the relation $[\Lambda, L] = h$, we have $< L\alpha, L\beta >=< \Lambda L\alpha, \beta >=< h\alpha + L\Lambda\alpha, \beta >=< h\alpha, \beta >= c < \alpha, \beta >$ where c is a constant. Hence if $k > s$, $< L^k\alpha, L^s\beta >= c < L^{k-s}\alpha, \beta >= c < L^{k-s-1}\alpha, \Lambda\beta >= 0$. Hence the decomposition is orthogonal with respect to $<,>$. (4) and (5) follow from some simple calculations. □

Definition Let $V = \oplus_{t=0}^{2n} H^{2t}$ be a Lefschetz algebra. Suppose that V is endowed with the following structures:

1. Each $H^{2t}_{\mathbb{C}} = \oplus_{p+q=2t} H^{p,q}$ has a Hodge structure of weight $2t$ such that the decomposition is orthogonal with respect to $<,>$.

2. The Hodge structure is compatible with the $sl_2(\mathbb{C})$-action, i.e.,

$$L^k : H^{p,q} \to H^{p+k,q+k}$$

   for any p, q, k.

3. Let $B^{p,q} := ker\Lambda : H^{p,q} \to H^{p-1,q-1}$ and define

$$Q(\alpha, \beta) :=< L^{n-r}\alpha, L^{n-r}\overline{\beta} >$$

   for $\alpha \in B^{p,q}, \beta \in B^{q,p}$ and $2r = p + q$. There are the Hodge-Riemann bilinear relations:

   (a) $Q(B^{p,q}, B^{s,t}) = 0$ if $s \neq q$.

   (b) $(-1)^{r+q}Q(\xi, \overline{\xi}) > 0$ if $0 \neq \xi \in B^{p,q}$ where $2r = p + q$.

Then we say that V is a Hodge-Lefschetz algebra.

From now on, our V denote a Hodge-Lefschetz algebra as above. Since the $sl_2(\mathbb{C})$-action is compatible with the Hodge structure, it reduces to an $sl_2(\mathbb{C})$-action on $V^{a,b}_{\mathbb{C}} = \oplus_{k=-min(a,b)}^{min(a,b)} H^{a+k,b+k}$. Hence for all p, q, we have a Lefschetz decomposition

$$H^{p,q} = B^{p,q} \oplus LB^{p-1,q-1} \oplus L^2B^{p-2,q-2} \oplus \cdots \oplus L^rB^{p-r,q-r}$$

where $r = min(p, q)$. Similar to the proof of Proposition 6.1, we have the orthogonality of the decomposition.

Proposition 6.2. *The decomposition*

$$H^{2t}_{\mathbb{C}} = \bigoplus_{p+q=2t} \bigoplus_{0\leq k\leq min(p,q)} L^kB^{p-k,q-k}$$

is orthogonal with respect to $<,>$.

Let $h^{p,q} = dim_{\mathbb{C}} H^{p,q}_{\mathbb{C}}$. We follow [12, Theorem 15.8.2] to give a proof of the Hodge index theorem.

Theorem 6.3. *(Abstract Hodge index theorem) Suppose that V is a Hodge-Lefschetz algebra as above. Define $(\alpha,\beta) :=< \alpha, \overline{*}\beta >$ on H^{2n}. Then the signature σ of $(\ ,\ )$ is $\sum_{p,q}(-1)^q h^{p,q}$.*

Proof. 1. Since $\overline{*}_{|V}$ is self-adjoint and $<,>_{|V\times V}$ is symmetric, $(\ ,\ )$ is a symmetric bilinear form.

2. Let $E_k^{p,q}$ be the vector space consisting of $L^k(a+\overline{a})$ for $a \in B^{p-k,q-k}$. Then $E_k^{p,q}$ is a real vector space. We have
$$H^{2n}_{\mathbb{R}} = \bigoplus_{p+q=2n} \bigoplus_{0\le k\le min(p,q)} E_k^{p,q}$$

3. The decomposition above is orthogonal with respect to the Hodge inner product and the quadratic form $(-1)^{q+k}(\ ,\ )$ is positive definite when restricted to $E_k^{p,q}$.

   Proof. For $\alpha \in E_k^{p,q}$ where $p+q=2n$ and $p\neq q$, let $\alpha = L^k a$ where $a = b+\overline{b}$ is real, $b \in B^{p-k,q-k}$, by a simple calculation, we have $\overline{*}L^k b = (-1)^{n-k}L^k\overline{b}$ then $(\alpha,\alpha) = 2 < L^k b, \overline{*}L^k b >= 2(-1)^{n-k} < L^k b, L^k b >= 2(-1)^{n-k}Q(b,\overline{b})$. Hence by the Hodge-Riemann bilinear relation, $(-1)^{q+k}(b,b) = 2(-1)^{(n-k)+(q-k)}Q(b,\overline{b}) > 0$ if $b\neq 0$. Similarly, if $p=q$, let $\alpha = L^k b$ where $b \in B^{n-k,n-k}$ and $b=\overline{b}$. Then $(\alpha,\alpha) =< \alpha, *\alpha >= (-1)^{n+k} < L^k b, L^k b >= (-1)^{n+k}Q(b,\overline{b})$. Hence, $(-1)^{n+k}(\alpha,\alpha) = Q(b,\overline{b}) > 0$ if $b\neq 0$. □

4. Therefore,
$$\sigma = \sum_{\substack{p+q=2n\\ k\le p\le q}} (-1)^{q+k} dim_{\mathbb{R}} E_k^{p,q}$$

5. $$\sigma = \sum_{\substack{p+q=2n\\ k\le min(p,q)}} (-1)^{q+k} dim_{\mathbb{C}} L^k B^{p-k,q-k}$$

   Proof. The real dimension of $E_k^{p,q}$ is $dim_{\mathbb{C}}L^kB^{p-k,q-k} + dim_{\mathbb{C}}L^kB^{p-k,q-k}$ for $p<q$ and $dim_{\mathbb{R}}E_k^{n,n} = dim_{\mathbb{C}}L^kB^{n-k,n-k}$. □

6. $h^{p-k,q-k} - h^{p-k-1,q-k-1} = dim_{\mathbb{C}}B^{p-k,q-k} = dim_{\mathbb{C}}L^kB^{p-k,q-k}$ for $p+q\le 2n$.

7. For $p+q=2n$, by the Hard Lefschetz theorem, we have $h^{p-k-1,q-k-1} = h^{p+k+1,q+k+1}$ and from the Hodge structures, $h^{r,k} = h^{k,r} = h^{2n-r,2n-k}$.

8. $$\sigma = \sum_{\substack{k\ge 0\\ p+q=2n}} (-1)^{q-k}h^{p-k,q-k} + \sum_{\substack{k\ge 0\\ p+q=2n}} (-1)^{q+k+1}h^{p+k+1,q+k+1}$$
$$= \sum_{p+q\le 2n} (-1)^q h^{p,q} + \sum_{p+q>2n} (-1)^q h^{p,q} = \sum_{p,q}(-1)^q h^{p,q}$$

□

7 Morphic Hodge Index Theorem

Let us use $H^*(X;\mathbb{C})$ to denote the cohomology ring of X. Let $h = \sum_{k=0}^{2m}(m-k)Pr_k$ where $Pr_k : H^*(X;\mathbb{C}) \longrightarrow H^k(X;\mathbb{C})$ projects a form to its k-component. The $sl_2(\mathbb{C})$-structure on $H^*(X;\mathbb{C})$ is given by

$$[\Lambda, L] = h, [h, \Lambda] = 2\Lambda, [h, L] = -2L$$

Let $\mathscr{A}$ be any of $EH^a(X;\mathbb{C}), OH^b(X;\mathbb{C})$ or $LH^{a,b}(X;\mathbb{C})$ and

$$\gamma(\mathscr{A}, p, q) = \begin{cases} a + \frac{p+q}{2}, & \text{if } p+q \text{ is even} \\ b + \frac{p+q-1}{2}, & \text{if } p+q \text{ is odd .} \end{cases}$$

If it is clear from the context what $\mathscr{A}$ is, to simplify our notation, we will just write $\gamma(p,q)$ for $\gamma(\mathscr{A}, p, q)$. Let $\mathcal{L}$ be the restriction of L to $\mathscr{A}$ and λ be the adjoint of $\mathcal{L}$ with respect to the Hodge inner product restricted to $\mathscr{A}$.

Proposition 7.1. *Assume that the morphic conjecture holds on $\mathscr{A}$, then $\mathscr{A}$ is a $sl_2(\mathbb{C})$-submodule of $H^*(X;\mathbb{C})$ thus*

1. *$\mathscr{A}$ has a sub-Lefschetz decomposition, i.e., if $\widetilde{L}^t H^k(X;\mathbb{C})$ is a direct summand of $\mathscr{A}$, then*

   $$\widetilde{L}^t H^k(X;\mathbb{C}) = \oplus_{r \geq max\{0,k-m\}} \mathcal{L}^r B^{m-2r}$$

   where $B^k = Ker\mathcal{L}^{m-k+1} : \widetilde{L}^t H^k(X;\mathbb{C}) \longrightarrow \widetilde{L}^{t+m-k+1} H^{m-k+2}(X;\mathbb{C})$ is the primitive group. Furthermore, this decomposition is compatible with the sub-Hodge structure, i.e., if $B^{p,q} = Ker L^{m+1-p-q} : H^{p,q}_{\gamma(p,q)}(X) \longrightarrow H^{m+1-q,m+1-p}_{\gamma(m+1-q,m+1-p)}(X)$, then

   $$H^{p,q}_{\gamma(p,q)}(X) = \oplus_{r \geq max\{0,k-m\}} \mathcal{L}^r B^{p-r,q-r}$$

2. *$B^k = ker\lambda : \widetilde{L}^t H^k(X;\mathbb{C}) \longrightarrow \widetilde{L}^{t-1} H^{k-2}(X;\mathbb{C})$ and $B^{p,q} = ker\lambda : H^{p,q}_t(X) \longrightarrow H^{p-1,q-1}_{t-1}(X)$ where $\widetilde{L}^t H^k(X;\mathbb{C})$ is a direct summand of $\mathscr{A}$.*

3. *We have the Hard Lefschetz theorem, i.e.,*

   $$\mathcal{L}^k : \widetilde{L}^t H^{m-k}(X;\mathbb{C}) \longrightarrow \widetilde{L}^{t+k} H^{m+k}(X;\mathbb{C})$$

   is an isomorphism where $\widetilde{L}^t H^{m-k}(X;\mathbb{C})$ is a direct summand of $\mathscr{A}$.

4. *We have the Hodge-Riemann bilinear relations:*

   (a) *$Q(B^{p,q}, B^{s,t}) = 0$ if $s \neq q$.*

   (b) *$(\sqrt{-1})^{-r}(-1)^q Q(\xi, \bar{\xi}) > 0$ if $0 \neq \xi \in B^{p,q}$ and $p+q = r$*

   where

   $$Q(\tau, \eta) = (-1)^{\frac{r(r+1)}{2}} \int_X \mathcal{L}^{n-r}(\tau \wedge \eta)$$

   and $\tau, \eta \in B^r$.

Proof. By the assumption of the morphic conjecture, λ is the restriction of Λ on $\mathscr{A}$. From the relation $h = [\Lambda, L]$, we see that h restricts to an operator on $\mathscr{A}$. Thus $\mathcal{L}, \lambda, h$ give a sub-$sl_2(\mathbb{C})$-structure on $\mathscr{A}$ and therefore it admits a sub-Lefschetz decomposition of the Lefschetz decomposition of $H^*(X;\mathbb{C})$ which is compatible with the sub-Hodge structure. The Hard Lefschetz theorem is a formal consequence of the Lefschetz decomposition (see e.g. [18], Chapter 11). The restriction of the classical Hodge-Riemann bilinear relations to $\mathscr{A}$ gives the similar relations. ∎

We observe that the morphic signatures are independent of the odd part $OH^b(X;\mathbb{C})$ of the cohomology groups. In the following, by assuming the morphic conjecture on $EH^a(X;\mathbb{C})$, we are going to generalize the classical Hodge index theorem.

Theorem 7.2. *(Hodge index theorem) If the morphic conjecture is true on $EH^a(X;\mathbb{C})$ where X is a connected projective manifold of dimension $m = 2n$ and a is a nonnegative integer, then*

$$\sigma_{a+n}(X) = \sum_{0\le p,q\le m} (-1)^q h^{p,q}_{\gamma(a,p,q)}$$

where $\gamma(a,p,q) = \gamma(EH^a(X;\mathbb{C}),p,q)$.

Proof. Let ω be the $(1,1)$-form associated to the standard Kähler metric on X and let $\mathscr{H}^{p,q}$ be the space of harmonic (p,q)-forms on X. The Lefschetz operator $L : \mathscr{H}^{p,q} \longrightarrow \mathscr{H}^{p+1,q+1}$ is defined by $L\alpha = \omega \wedge \alpha$.

Since ω is integral, L is an operator on $EH^a(X;\mathbb{Q})$. We have the Hodge inner product $<,>$ on $EH^a(X;\mathbb{C})$, the adjoint operator Λ of L, the Hodge star operator $\bar{*} : \mathscr{H}^{p,q} \longrightarrow \mathscr{H}^{n-p,n-q}$, and the cup product pairing $(\ ,\ )$ on $H^m(X;\mathbb{C})$ satisfying $(\alpha,\beta) =<\alpha,\bar{*}\beta>$ for all $\alpha,\beta \in H^m(X;\mathbb{C})$.

By the sub-Hodge structure on $\widetilde{L}^sH^k(X;\mathbb{C})$, we decompose $\widetilde{L}^sH^k(X;\mathbb{C}) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}_s$. Now we assume that the morphic conjecture is true on $EH^a(X;\mathbb{C})$. Then $EH^a(X;\mathbb{C})$ is a Hodge-Lefschetz algebra. The result now follows from Theorem 6.3. $\square$

Corollary 7.3. *When $a = n$, $h^{p,q}_{\gamma(n,p,q)} = h^{p,q}$, the above formula gives the classical Hodge index theorem:*

$$\sigma(X) = \sigma_{2n}(X) = \sum_{0\le p,q\le m} (-1)^q h^{p,q}_{\gamma(n,p,q)} = \sum_{0\le p,q\le m} (-1)^q h^{p,q}.$$

Example

1. Suppose that X is a complex projective surface. Then $\sigma_1(X) = 2-h^{1,1}(X), \sigma_2(X) = \sigma(X) = 2 + 2h^{2,0}(X) - h^{1,1}(X)$.

2. Suppose that X is a general polarized abelian variety of dimension $g = 2n$. By a theorem of Mattuck, we have $H^{p,p}(X;\mathbb{Q}) \simeq \mathbb{Q}$ for $0 \le p \le g$ (see [1], pg 559). Thus $\sigma_n(X) = 1$. But as a smooth manifold, X is the boundary of a solid torus, hence the signature of X is 0.

3. For a smooth hypersurface X of dimension $m = 2n$ in $\mathbb{P}^{2n+1}$, for $p \ne n$, $H^{p,p}(X;\mathbb{Q})$ is 1-dimensional and is generated by algebraic cycles. Therefore the adjoint operator

$$\Lambda : H^{2n-p,2n-p}(X;\mathbb{Q}) \longrightarrow H^{p,p}(X;\mathbb{Q})$$

is an isomorphism for $0 \le p < n$. For $p = n$, $\Lambda : H^{n,n}(X;\mathbb{Q}) \longrightarrow H^{n,n}(X;\mathbb{Q})$ is an isomorphism. The GSCA is trivially true for this case, hence the signature formula

$$\sigma_{a+n}(X) = 1 + (-1)^{n-1} + (-1)^n h^{n,n}_{a+n}$$

is valid. In particular, we are especially interested in

$$\sigma_n(X) = 1 + (-1)^{n-1} + (-1)^n h^{n,n}_n$$

where $h^{n,n}_n$ is the dimension of the subspace of $H^{n,n}(X)$ which is generated by algebraic cycles. Thus any way to calculate $\sigma_n(X)$ is equivalent to the calculation of $h^{n,n}_n$. The Hodge conjecture predicts that $h^{n,n}_n = h^{n,n}_{\mathbb{Q}}$ where $h^{n,n}_{\mathbb{Q}}$ is the dimension of $H^{n,n}(X;\mathbb{Q}) := H^{n,n}(X) \cap H^{2n}(X;\mathbb{Q})$.

We note that even for smooth hypersurfaces of even dimension, the Hodge conjecture is known only for some small degree.

Let $\sigma_n(X) = a - b$ the difference between the numbers of positive and negative eigenvalues of $(\ , \ )$. By the Poincaré duality theorem, $(\ , \ )$ is non-degenerate, hence $a + b = h_n^{n,n}$ and we get $a = \frac{(-1)^n+1}{2} h_n^{n,n}$. Therefore, the cup product pairing $(\ , \ )$ is positive definite on $H_n^{n,n}$ if n is even and negative definite on $H_n^{n,n}$ if n is odd.

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